Chapter V
Global Stability Analysis for Complex-Valued Recurrent Neural Networks and Its Application to Convex Optimization Problems

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ABSTRACT
Global stability analysis for complex-valued artificial recurrent neural networks seems to be one of yet-unchallenged topics in information science. This chapter presents global stability conditions for discrete-time and continuous-time complex-valued recurrent neural networks, which are regarded as nonlinear dynamical systems. Global asymptotic stability conditions for these networks are derived by way of suitable choices of activation functions. According to these stability conditions, there are classes of discrete-time and continuous-time complex-valued recurrent neural networks whose equilibrium point is globally asymptotically stable. Furthermore, the conditions are shown to be successfully applicable to solving convex programming problems, for which real field solution methods are generally tedious.

INTRODUCTION
Recurrent neural networks whose neurons are fully interconnected have been utilized to implement associative memories and solve optimization problems. These networks are regarded as nonlinear dynamical feedback systems. Stability properties of this class of dynamical networks are an important issue from applications point of view.
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On the other hand, several models of neural networks that can deal with complex numbers, the complex-valued neural networks, have come to forth in recent years. These networks have states, connection weights, and activation functions, which are all complex-valued. Such networks have been studied in terms of their abilities of information processing, because they possess attractive features which do not exist in their real-valued counterparts (Hirose, 2003; Kuroe, Hashimoto & Mori, 2001, 2002; Kuroe, Yoshida & Mori, 2003; Nitta, 2000; Takeda & Kishigami, 1992; Yoshida, Mori & Kuroe, 2004; Yoshida & Mori, 2007). Generally, activation functions of neural networks crucially determine their dynamic behavior. In complex-valued neural networks, there is a greater choice of activation functions compared to real-valued networks. However, the question of appropriate activation functions has been paid insufficient attention to in the past.

Local asymptotic stability conditions for complex-valued recurrent neural networks with an energy function defined on the complex domain have been studied earlier and synthesis of complex-valued associative memories has been realized (Kuroe et al., 2001, 2002). However, studies on their application to global optimization problems and theoretical analysis for global asymptotic stability conditions remain yet-unchallenged topics.

The purpose of this chapter is to analyze global asymptotic stability for complex-valued recurrent neural networks. Two types of complex-valued recurrent neural networks are considered: discrete-time model and continuous-time model. We present global asymptotic stability conditions for both models of the complex-valued recurrent neural networks. To ensure global stability, classes of complex-valued functions are defined as the activation functions, and thereafter several stability conditions are obtained. According to these conditions, there are classes of discrete-time and continuous-time complex-valued recurrent neural networks whose common equilibrium point is globally asymptotically stable. Furthermore, the obtained conditions are shown to be successfully applicable to solving convex programming problems.

The chapter is organized as follows. In Background, a brief summary of applications to associative memories and optimization problems in real-valued recurrent neural networks is presented. Moreover, results on stability analysis and applications of these real-valued neural networks are introduced. Next, models of discrete-time and continuous-time complex-valued neural networks are described. For activation functions of these networks, two classes of complex-valued function are defined. In the next section, global asymptotic stability conditions for the discrete-time and continuous-time complex-valued recurrent neural networks are proved, respectively. Some discussions thereof are also given. Furthermore, applications of complex-valued neural networks to convex programming problems with numerical examples are shown in the subsequent section. Finally, concluding remarks and future research directions are given.

Before going into the body of the chapter, we first list the glossary of symbols. In the following, the sets of \( n \times m \) real and complex matrices are defined by \( \mathbb{R}^{n \times m}, \mathbb{C}^{n \times m} \), respectively. \( \mathbf{I}_n \) denotes the identity matrix in \( \mathbb{R}^{n \times n} \). \( \mathbb{R}_+ \) means the nonnegative space in \( \mathbb{R} \) defined by \( \mathbb{R}_+ = \{ x \mid x \in \mathbb{R}, x \geq 0 \} \). For a complex number \( x \in \mathbb{C}, |x| \) stands for the absolute value, and \( \overline{x} \) is the complex conjugate number. \( \text{Re}(x) \) denotes the real part of \( x \in \mathbb{C} \), and \( \text{Im}(x) \) denotes the imaginary part of \( x \in \mathbb{C} \). For any pair of complex numbers \( x, y \in \mathbb{C} \), \( \langle x, y \rangle \) denotes the inner product defined by \( \langle x, y \rangle = \overline{y}^T x \). For a complex matrix \( X \in \mathbb{C}^{n \times m} \) represented by \( x = \{ x_i \} \), \( X^\ast \) and \( X^T \) denote the transpose and conjugate transpose, respectively. If \( X \in \mathbb{C}^{n \times n} \) is a Hermitian matrix \( (X = X^T) \), \( X > 0 \) denotes that \( X \) is positive definite. \( \lambda_{\min}(X) \) and \( \lambda_{\max}(X) \) represent the minimum and the maximum eigenvalue of a Hermitian matrix \( X \), respectively. \( |X| \) represents the element-wise absolute-value matrix defined by \( |X| = \{ |x_i| \} \), and \( \|X\|_2 \) is the induced matrix 2-norm defined by \( \|X\|_2 = \sqrt{\lambda_{\max}(X^T X)} \). Suppose that \( X \) is an \( n \times n \) real matrix with nonnegative off-diagonal elements, then \( X \) is a nonsingular M-matrix if and only if all principal minors of \( X \) are positive.

BACKGROUND

Proposals of models for neural networks and its applications by Hopfield et al. have triggered the research interests of neural networks in the last two decades (Hopfield, 1984; Hopfield & Tank, 1985; Tank & Hopfield, 1986). They introduced the idea of an energy function to formulate a way of understanding the computational ability that performed by fully connected recurrent neural networks. The energy functions have been applied to vari-
ous problems such as qualitative analysis of neural networks, synthesis of associative memories, combination optimization problems, and linear programming problems.

When used as associative memories, neural networks are designed so that memory pattern vectors correspond to locally stable equilibrium points of the networks. To realize that an imperfect input pattern matches a correct stored pattern, local stability and attracting region of the equilibrium points become important issues. Such stability analysis and synthesis of associative memories using neural networks have been studied actively in the past (see, for example, Li, Michel & Porod, 1988).

In case of optimization problems, the neural networks are constructed in such a way that locally or globally optimal solutions correspond to equilibrium points of the networks. To obtain optimal solutions, it is desired that the state evolution of the networks converges to the equilibrium points independent of the initial conditions. Especially, global asymptotic stability guarantees to find the global optimal solution, which corresponds to the sole equilibrium. This prompted global asymptotic stability analysis for the networks, which has been intensively carried out in recent years. In the following, some major results on global stability analysis for discrete-time and continuous-time real-valued neural networks are introduced.

For the discrete-time recurrent neural networks with the activation functions which are belong to the class of \textit{sigmoid} functions, functions that are bounded, continuously differentiable and strictly monotone increasing, global asymptotic stability conditions are derived as Lyapunov matrix inequality form (Jin & Gupta, 1996). For larger class of activation functions that are globally Lipschitz continuous and monotone non-decreasing, global asymptotic and exponential stability conditions are shown (Hu & Wang, 2002).

On the other hand, for the continuous-time recurrent neural networks, when it is assumed that the activation functions belong to the class of \textit{sigmoid} functions, and the connection weight matrix is symmetric, a necessary and sufficient condition for global asymptotic stability of the networks is formulated as negative semi-definiteness of the connection weight matrix (Forti, Manetti & Marini, 1994). If activation functions are unbounded, monotone non-decreasing and globally Lipschitz continuous, a sufficient condition for global asymptotic stability is derived as the form of Lyapunov matrix inequality for the connection weight matrix and the Lipschitz numbers (Forti & Tesi, 1995). Furthermore, with the same class of activation functions, a sufficient condition of global exponential stability is presented (Liang & Si, 2001). For general class of activation functions, that is, unbounded and monotone non-decreasing functions, Lyapunov matrix inequality and row or column dominance conditions for connection weight matrix are presented (Arik & Tavsanoglu, 2000; Forti & Tesi, 1995).

For applications of the real-valued recurrent neural networks, Hopfield applied to linear programming problems (Tank & Hopfield, 1986). Moreover, for extension of this application, applications to upper and lower bounded constrained quadratic programming problems (Bouzerdoum & Pattison, 1993), linearly constrained quadratic programming problems (Maa & Shanblatt, 1992) and nonlinear optimization problems (Kennedy & Chua, 1988) are presented. Furthermore, applications to general class of optimization problems, that is, variational inequality problems including convex programming problems and complementarity problems are investigated with global asymptotic or exponential stability conditions (Liang & Si, 2001; Xia, & Wang, 2000, 2001, 2004, 2005).

In this chapter, we explore the line of the above-mentioned investigations, global stability analysis and applications, for complex-valued recurrent neural networks.

\section*{COMPLEX-VALUED NEURAL NETWORKS}

This section presents models of discrete-time and continuous-time recurrent neural networks whose states, input and output variables, and connection weight coefficients are all complex-valued. The discrete-time complex-valued recurrent neural network is described by difference equations of the form:

\begin{equation}
\begin{bmatrix}
u[k+1] \\
v[k]
\end{bmatrix} = \begin{bmatrix}Au[k] + Wv[k] + b \\
f(u[k])
\end{bmatrix}.
\end{equation}

While, the continuous-time complex-valued recurrent neural network is given by differential equations of the form:
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\[ \begin{cases} \frac{du(t)}{dt} = -Du(t) + Wv(t) + b \\ v(t) = f(u(t)) \end{cases} \]  \hspace{1cm} (2)

In these models, \( n \) is a number of neuron units, \( u = [u_1, u_2, \cdots, u_n] \in \mathbb{C}^n \) and \( v = [v_1, v_2, \cdots, v_n] \in \mathbb{C}^n \) are the state and output vector respectively, \( b = [b_1, b_2, \cdots, b_n] \in \mathbb{C}^n \) is the external input vector, and \( W = \{w_{ij}\} \in \mathbb{C}^{n \times n} \) is the connection weight matrix.

\( A = \text{diag}[a_1, a_2, \cdots, a_n] \in \mathbb{R}^{n \times n} \) with \(|a_i| < 1 \) \( i = 1, 2, \cdots, n \) and \( D = \text{diag}[d_1, d_2, \cdots, d_n] \in \mathbb{R}^{n \times n} \) with \( d_i > 0 \) \( i = 1, 2, \cdots, n \)

specify the local feedbacks around each neuron. \(|a_i| < 1 \) and \( d_i > 0 \) indicate self-inhibitions. \( f(u) = [f_1(u_1), f_1(u_2), \cdots, f_1(u_n)] : \mathbb{C}^n \rightarrow \mathbb{C}^n \) is the vector-valued activation function whose elements consist of complex-valued nonlinear functions.

The activation functions \( f_i(.) : \mathbb{C} \rightarrow \mathbb{C} \) will be assumed to belong to the following classes of complex-valued functions.

**Definition 1.** A set of complex-valued functions \( f_i(i = 1, 2, \cdots, n) \) satisfying the following conditions is said to be class \( F^m \).

1. \( f_i(0) = 0 \),
2. \( f_i \) is bounded,
3. there exists a positive real value \( l > 0 \) such that, \(|f_i(x) - f_i(y)| \leq l |x - y| \) \( \forall x, y \in \mathbb{C} \)
4. \( \Re \{l f_i(x) - f_i(y)\} \geq 0 \) \( \forall x, y \in \mathbb{C} \).

**Definition 2.** A set of complex-valued functions \( f_i(i = 1, 2, \cdots, n) \) is said to be class \( S \) if its member functions are represented by:

\[ f_i(u_i) = \varphi_i(|u_i|) \frac{u_i}{|u_i|} \]  \hspace{1cm} (3)

with nonnegative real-valued function \( \varphi_i(r_i) : \mathbb{R}_+ \rightarrow \mathbb{R}_+ \) which satisfies the following conditions.

1. \( \varphi_i(0) = 0 \),
2. \( \varphi_i \) is bounded,
3. there exists a continuous derivative function \( \frac{d \varphi_i}{dr_i} \) in \( \mathbb{R}_+ \) ,
4. there exists a positive real value \( l > 0 \) such that \( \frac{d \varphi_i}{dr_i}(\xi) \leq l < +\infty \) \( \forall \xi \in \mathbb{R}_+ \).

Note that, in Definition 2, because \( u_i / |u_i| \) is bounded, \( f_i(0) = 0 \). The following properties about the class \( S \) will be utilized for proving stability theorem in the next section.

**Property 1.** Let us define a complex-valued function as \( h_i(u_i) := f_i(u_i + u_i') - f_i(u_i') : \mathbb{C} \rightarrow \mathbb{C} \) with an arbitrary fixed complex number \( u_i' \in \mathbb{C} \).

If \( h_i(u_i) \in S \), then \( \frac{\partial \Re(h)}{\partial \Im(u_i)} = \frac{\partial \Im(h)}{\partial \Re(u_i)} \).

**Property 2.** If \( f_i(.) \in S \), then

\[ \Re \{l f_i(x) - f_i(y)\} \geq l^{-1} |f_i(x) - f_i(y)|^2 \]  \hspace{1cm} (4)
Property 1 can be immediately derived by calculating the partial derivatives as follows:

\[
\frac{\partial \text{Re}(h_j)}{\partial \text{Im}(u_i)} = \frac{\partial \text{Im}(h_j)}{\partial \text{Re}(u_i)} = \left( \frac{d\varphi_i}{dr_j} \left( |u_i| \right) - \frac{\varphi_i'(|u_i|)}{|u_i|} \right) \frac{\text{Re}(u_i) \text{Im}(u_i)}{|u_i|^2}.
\]

The proof of Property 2 is given in Appendix. Regarding a complex-valued function as 2-dimensional vector-valued map, Property 1 means that \( f \) is a kind of symmetric map, and Property 2 means a globally Lipschitz continuous and monotone non-decreasing map.

According to the Liouville’s theorem (Churhill & Brown, 1984, p. 119), every bounded holomorphic function must be constant. It is therefore assumed that \( f_i \in F^{im} \) or \( f_i \in S \) is not a holomorphic function throughout this chapter to avoid triviality. When a complex-valued function is regarded as a 2-dimensional real vector-valued function, the condition 4 in Definition 1 and Definition 2 imply that the mappings \( f_i \in F^{im} \) and \( f_i \in S \) are monotone. We note that the monotonic function is in general suitable for formulation of global optimization problems.

As a specific example of the functions which belong to the class \( F^{im} \),

\[
\psi_{K_i}(u_i) = \arg \min \left\{ |u_i - z_i| \mid z_i \in K_i \right\}
\]

will be considered with \( K_i \subset \mathbb{C} \) being any bounded, closed and convex set including the origin. This function is said to be Convex projection in optimization theory (Fukushima, 2001), and has been often used when real-valued neural networks are applied to solve convex programming problems (Xia & Wang, 2004). In addition, the following functions are typical to the class \( F^{im} \) or \( f_i \in S \) and are often used in the analysis and design with the complex-valued neural networks (Hirose, 2003; Kuroe et al., 2001, 2002; Kuroe et al., 2003; Takeda & Kishigami, 1992; Yoshida et al., 2004),

\[
f_i(u_i) = \tanh\left( |u_i| \right)\frac{u_i}{|u_i|}, \quad (6)
\]

\[
f_i(u_i) = \frac{u_i}{1 + |u_i|}, \quad (7)
\]

If all the activation functions of complex-valued recurrent neural networks satisfy that \( f_i \in F^{im} \) or \( f_i \in S \) \((i = 1, 2, \cdots, n)\), there exists a positive diagonal matrix \( L = \text{diag}\{l_1, l_2, \cdots, l_n\} \in \mathbb{R}^{n \times n} \) from the condition 3 of Definition 1 and the condition 4 of Definition 2. Particularly, if \( f_i \) is selected as the form of Eq. (5), then \( L = I_n \).

This property will be used in later section: Application to convex programming problems. The matrix \( L \) will be used to derive stability conditions in the next section.

**GLOBAL ASYMPTOTIC STABILITY CONDITIONS**

This section provides global asymptotic stability conditions for the discrete-time and continuous-time complex-valued neural networks with the activation functions \( f_i \in F^{im} \) or \( f_i \in S \).

**Discrete-Time Neural Networks**

For the discrete-time neural network (Eq. (1)), the following theorem ensures global asymptotic stability of the network under the condition expressed in terms of the connection weight matrix.
Theorem 1. Suppose that all the activation functions satisfy $f_i \in P^m$ for the discrete-time complex-valued neural network (Eq. (1)). The network (Eq. (1)) has a unique globally asymptotically stable equilibrium point, if there exists a positive definite diagonal matrix $P = \text{diag}\{p_1, p_2, \cdots, p_n\} \in \mathbb{R}^{n \times n}$ such that

$$\left(\frac{\gamma}{1+\gamma} P - \gamma A^* PA\right) L^2 - |W|^2 |P|W > 0$$

with the diagonal matrix given by $L = \text{diag}\{l_1, l_2, \cdots, l_n\} \in \mathbb{R}^{n \times n}$ and the real positive number $\gamma$.

Proof: The equilibrium points are nothing but the fixed points of the continuous map $\bar{\chi}(u) := (I_n - A)^{-1}(Wf(u) + b)$. According to Brower’s fixed-point theorem (Luenberger, 1969, p. 272; Nakaoka, 1977, pp. 7-14), if the continuous map $\bar{\chi}(u)$ is bounded, then $\bar{\chi}(u)$ has at least one fixed point. Since all the activation functions of the network (Eq. (1)) are bounded, the network has at least one equilibrium point.

Let us represent one of the equilibrium points by $u^* \in \mathbb{C}^n$. By means of the coordinate shift $z = u - u^*$, Eq. (1) can be put into the form:

$$z[k+1] = Az[k] + Wh(z[k])$$
$$h(z[k]) = f(z[k] + u^*) - f(u^*)$$.

Thus, Eq. (9) has an equilibrium point at the origin, i.e. $z = 0$ and we therefore focus on the stability property of the origin. To prove that $z = 0$ is globally asymptotically stable, consider a candidate Lyapunov function of the quadratic form:

$$V_1(z[k]) = z[k]^* Pz[k]$$

where $P$ is a positive definite diagonal matrix which satisfies the condition of Theorem 1. In the following, we simply write $z = z[k]$ for the sake of convenience. The forward difference $\Delta V_1$ of $V_1$ along the trajectory of Eq. (9) is calculated as follows:

$$\Delta V_1(z)\bigg|_{(9)} = z[k+1]^* Pz[k+1] - z[k]^* Pz[k]$$
$$= -z^* (P - A^* PA)z + 2 \text{Re} \left\{z^* A^* PWh(z) \right\} + (z^* W^* PWh(z))$$

where $\Delta V_1(z)\bigg|_{(9)}$ denotes the difference along the trajectories of Eq. (9). Here we introduce free parameter $\gamma$ satisfying the condition of Theorem 1. By expanding the obvious inequality,

$$\left\| P^{1/2} \left\{ \gamma^{1/2} Az - \gamma^{-1/2} Wh(z) \right\} \right\|^2 \geq 0,$$

we have

$$2 \text{Re} \left\{ z^* A^* PWh(z) \right\} \leq \gamma z^* A^* PAz + \gamma^{-1} h^*(z) W^* PWh(z)$$

with $P^{1/2} = \text{diag}\{p_1^{1/2}, p_2^{1/2}, \cdots, p_n^{1/2}\}$. Hence,

$$\Delta V_1(z)\bigg|_{(9)} \leq -z^* \left\{ P - (1+\gamma) A^* PA \right\} z + (1+\gamma^{-1}) h^*(z) W^* PWh(z)$$

From the condition 3 of Definition 1, it follows that

$$\Delta V_1(z)\bigg|_{(9)} \leq -z^* \left\{ P - (1+\gamma) A^* PA \right\} z + (1+\gamma^{-1}) z^* L |W|^2 |P|W|L|z|$$
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\[-(1+\gamma^{-1})|z|^{\gamma}LRL|z| \]

where

\[R = \left( \frac{\gamma}{1+\gamma} P - \gamma A^*PA \right) L^{-2} - |W|^T P |W| \]

Since the matrix $R$ is positive definite due to Eq. (8), $\Delta V_1(z)_{(0)} < 0$, $z \neq 0$, and if $z = 0$, then $\Delta V_1(z)_{(0)} = 0$. Hence $z = 0$ that is, the equilibrium point $u^*$, is unique and asymptotically stable. Now, it is easy to see that the positive definite Lyapunov function (Eq. (10)) is radially unbounded, i.e. $\|V_1(z)\| \to \infty$ as $\|z\| \to \infty$. Therefore, the equilibrium point $u^*$ is globally asymptotically stable. (Q. E. D.)

In particular case, if $A = 0$, then the following corollary immediately follows by letting $A = 0$ and $\gamma \to \infty$ in the proof of Theorem 1.

Corollary 1. Suppose that all the activation functions satisfy $f_i \in F^{im}$ and $A = 0$ for the discrete-time complex-valued neural network (Eq. (1)). The equilibrium point of the network (Eq. (1)) is unique and globally asymptotically stable, if there exists a positive definite diagonal matrix $P = \text{diag}\{p_1, p_2, \cdots, p_n\} \in \mathbb{R}^{n \times n}$ such that

\[L^{-1}PL^{-1} - |W|^T P |W| > 0 \]

with the diagonal matrix given by $L = \text{diag}\{l_1, l_2, \cdots, l_n\} \in \mathbb{R}^{n \times n}$.

This corollary will be utilized in applications to convex programming problems in the next section.

Continuous-Time Neural Networks

For the continuous-time complex-valued recurrent neural network (Eq. (2)) with the activation functions which belong to the class $F^{im}$, the global asymptotic stability condition is derived as in the following theorem.

**Theorem 2.** Suppose that $f_i \in F^{im}$ for the continuous-time complex-valued neural network (Eq. (2)). The network (Eq. (2)) has a unique and globally asymptotically stable equilibrium point, if there exists a positive definite diagonal matrix $P = \text{diag}\{p_1, p_2, \cdots, p_n\} \in \mathbb{R}^{n \times n}$ such that

\[P (DL^{-1} - |W|) + (DL^{-1} - |W|)^T P > 0 \]

with the diagonal matrix given by $L = \text{diag}\{l_1, l_2, \cdots, l_n\} \in \mathbb{R}^{n \times n}$.

**Proof:** The equilibrium points correspond to the fixed points of the continuous map $\chi(u) := D^{-1}(Wf(u) + b)$. Similar to the proof of Theorem 1, the network has at least one equilibrium point.

Let us represent one of the existing equilibrium points by $u^* \in \mathbb{C}^n$. By the coordinate translation $z = u - u^*$, Eq. (2) can be put into the form:

\[
\frac{dz}{dt} = -Dz + Wh(z) \\
h(z) = f(z + u^*) - f(u^*).
\]

(15)

The network (Eq. (15)) has an equilibrium point $z = 0$. To prove that $z = 0$ is unique and globally asymptotically stable, consider a candidate Lyapunov function of the quadratic form:

\[V_1(z(t)) = z(t)^T Qz(t) \]

where $Q$ has the form of $Q = LP$ with $P$ and $L$ being positive definite diagonal matrices which satisfy the condition of Theorem 2 and the condition 3 of Definition 1, respectively. By regarding $V_1(z(t))$ as the real-valued
function $V_2(\text{Re}\{z(t)\}, \text{Im}\{z(t)\} \colon \mathbb{R} \to \mathbb{R}^{2n} \to \mathbb{R}$, the time derivative of $V_2$ along the trajectory of Eq. (15) is calculated as follows:

$$\frac{dV_2(z)}{dt}\bigg|_{(15)} = -2z^*QDz + 2\text{Re}\{z^*Q\text{Wh}(z)\}$$

$$= -2|z|^2QD|z| + 2\text{Re}\{z^*Q\text{Wh}(z)\}$$

$$\leq -2|z|^2QD|z| + 2|z|^2QW\|h(z)\|.$$  

(17)

Here we simply omit the argument of $z(t)$. From the condition 3 of Definition 1, it follows that

$$\frac{dV_2(z)}{dt}\bigg|_{(15)} \leq -2|z|^2QD|z| + 2|z|^2QW\|L\|z|$$

$$= -2|z|^2QD-\|W\|L\|z|$$

$$= -2|z|^2Q\left((D+1-\|W\|L)\right)z$$

$$= -2|z|^2L\left\{P(D+1-\|W\|) + (D+1-\|W\|)P)\right\}L\|z|.$$  

(18)

Since the matrix $P(D+1-\|W\|) + (D+1-\|W\|)P$ is positive definite because of Eq. (14), $dV_2(z)/dt\bigg|_{(15)} < 0$, $z \neq 0$, and if $z=0$, then $dV_2(z)/dt\bigg|_{(15)} = 0$. Hence $z=0$, that is, the equilibrium point $u^e$ of the network (Eq. (2)), is unique and asymptotically stable. Now, it is easy to see that the positive definite Lyapunov function (Eq. (16)) is radially unbounded, that is, $V_2(z) \to \infty$ as $\|z\| \to \infty$. This shows that the unique equilibrium point $u^e$ is globally asymptotically stable. (Q. E. D.)

When the activation functions belong to the class $S$, the global asymptotic stability condition for the network (Eq. (2)) is derived as in the following theorem.

**Theorem 3.** Suppose that $f_1 \in S$ for the continuous-time complex-valued neural network (Eq. (2)). The network (Eq. (2)) has a unique and globally asymptotically stable equilibrium point, if there exists a positive definite diagonal matrix $P = \text{diag}\{p_1, p_2, \cdots, p_n\} \in \mathbb{R}^{n \times n}$ such that

$$P(D+1-\|W\|) + (D+1-\|W\|)P > 0.$$  

(19)

with the diagonal matrix given by $L = \text{diag}\{l_1, l_2, \cdots, l_n\} \in \mathbb{R}^{n \times n}$.

**Proof:** The equilibrium points are equivalent to the fixed points of the continuous map $\chi(u) := D^{-1}(Wf(u) + b)$. Similar to the proofs of Theorem 1 and Theorem 2, the network has at least one equilibrium point.

Let us represent one of the existing equilibrium points by $u^e \in \mathbb{C}^n$. By the coordinate shifting $z = u - u^e$, Eq. (2) can be put into the form of Eq. (15). This has an equilibrium point $z = 0$. To prove that $z = 0$ is unique and globally asymptotically stable, consider a candidate Lyapunov function of the following form:

$$V_3(z(t)) = \frac{1}{2}z(t)^*D^{-1}z(t) + \sum_{i=1}^{n} p_i G_i(\text{Re}(z_i), \text{Im}(z_i))$$

(20)

where $P = \text{diag}\{p_1, p_2, \cdots, p_n\}$ is the positive definite diagonal matrix which satisfies Eq. (19), $\varepsilon$ and $k$ are constant parameters given by
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\[
\begin{align*}
\varepsilon \in (0, 1), \\
k = \frac{\|D^r W\|}{4\lambda_{\text{min}}(Q)} > 0, \\
2Q = P(\mathbf{DL}^{-1} - \mathbf{W}) + (\mathbf{DL}^{-1} - \mathbf{W})^\dagger P,
\end{align*}
\]

and \(G_i(\cdot)(i = 1, 2, \cdots, n)\) are defined by

\[
G_i(\text{Re}(z_i), \text{Im}(z_i)) = \int_0^{\text{Re}(z_i)} \text{Re}\{h_i(\rho + j0)\} d\rho + \int_0^{\text{Im}(z_i)} \text{Im}\{h_i(\text{Re}(z_i) + j\rho)\} d\rho
\]

where \(j\) is the imaginary unit. From Property 1 for the class \(S\), Eq. (22) can be transformed into the following form:

\[
G_i(\text{Re}(z_i), \text{Im}(z_i)) = \int_0^1 \text{Re}\{z_i h_i(\eta z_i)\} d\eta
\]

According to Property 2, Eq. (23) is nonnegative. Hence, \(V_3(\mathbf{z})\) is also nonnegative and \(V_3(\mathbf{z}) \geq \left(\frac{1}{2}\right)\mathbf{z}^* D^{-1} \mathbf{z}\).

By regarding \(V_3(\mathbf{z}(t))\) as the real-valued function \(V_3(\text{Re}(\mathbf{z}(t)), \text{Im}(\mathbf{z}(t))): \mathbb{R} \to \mathbb{R}^+ \to \mathbb{R}\), the time derivative of \(V_3\) along the trajectory of Eq. (15) is calculated as follows:

\[
\frac{dV_3(\mathbf{z})}{dt} \bigg|_{(15)} = \sum_{i=1}^n \frac{\partial V_3}{\partial \text{Re}(z_i)} \frac{d \text{Re}(z_i)}{dt} + \frac{\partial V_3}{\partial \text{Im}(z_i)} \frac{d \text{Im}(z_i)}{dt}
\]

\[
= \text{Re}\left\{\left[D^{-1}\mathbf{z} + \frac{k}{\varepsilon} P\mathbf{h}(\mathbf{z})\right]^* \frac{d \mathbf{z}}{dt}\right\}
\]

\[
= -\|\mathbf{z}\|^2 + \text{Re}\left\{\mathbf{z}^* D^{-1} P\mathbf{h}(\mathbf{z})\right\} - \frac{k}{\varepsilon} \text{Re}\left\{\mathbf{z}^* P\mathbf{Dh}(\mathbf{z}) - \mathbf{h}(\mathbf{z})^* P\mathbf{Wh}(\mathbf{z})\right\}
\]

where in the first equality, the following fact is used:

\[
\frac{dG_i}{dt} \bigg|_{(15)} = \frac{\partial G_i}{\partial \text{Re}(z_i)} \frac{d \text{Re}(z_i)}{dt} + \frac{\partial G_i}{\partial \text{Im}(z_i)} \frac{d \text{Im}(z_i)}{dt}
\]

\[
= \text{Re}\left\{h_i(\mathbf{z})^* \frac{d \mathbf{z}}{dt} + \text{Im}\{h_i(\mathbf{z})\} \frac{d \mathbf{z}}{dt}\right\}
\]

\[
= \text{Re}\left\{h_i(\mathbf{z}) \frac{dz_i}{dt}\right\}.
\]

We furthermore use the inequality \(\text{Re}\{\mathbf{z}^* P\mathbf{Dh}(\mathbf{z})\} \geq \mathbf{h}(\mathbf{z})^* P\mathbf{D}^{-1} \mathbf{h}(\mathbf{z})\), which comes from Eq. (4) in Property 2. By substituting this inequality, we have
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\[
\frac{dV(z)}{dt} \leq -\|z\|^2 + \text{Re}\left\{z^* D^{-1} Wh(z) \right\} - \frac{k}{\varepsilon} \text{Re}\left\{h(z)^* P D L^{-1} h(z) - h(z)^* P W h(z) \right\}
\]

\[
= -\|z\|^2 + \text{Re}\left\{z^* D^{-1} Wh(z) \right\} - \frac{k}{\varepsilon} h(z)^* Q h(z).
\]

Since \(\varepsilon \in (0, 1)\), by expanding \(\|z\|^2 \geq -1\), we obtain

\[
\text{Re}\left\{z^* D^{-1} Wh(z) \right\} \leq \varepsilon \|z\|^2 + \frac{\|D^{-1} Wh(z)\|^2}{4\varepsilon}.
\]

Hence, it follows that

\[
\frac{dV(z)}{dt} \leq -(1 - \varepsilon) \|z\|^2 + \left(\frac{\|D^{-1} W\|^2}{4\varepsilon} - \frac{k \lambda_{\text{max}}(Q)}{\varepsilon}\right) \|h(z)\|^2
\]

\[
= -(1 - \varepsilon) \|z\|^2.
\]

We now have \(dV(z)/dt = 0\), if \(z \neq 0\), and if \(z = 0\), then \(dV(z)/dt = 0\). Thus \(z = 0\), that is, the equilibrium point \(u^*\) of the network (Eq. (2)), is unique and asymptotically stable. Because \(D\) is positive definite diagonal matrix and \(G_i(\cdot, \cdot)\) \((i = 1, 2, \ldots, n)\) are positive, it is easy to see that the positive definite Lyapunov function (Eq. (20)) is radially unbounded, that is, \(V(z) \rightarrow \infty\) as \(\|z\| \rightarrow \infty\). This shows that the unique equilibrium point \(u^*\) is globally asymptotically stable. (Q. E. D.)

Discussions

Now, we compare Theorem 1 and Theorem 2 with some results of previous stability investigations for the real-valued recurrent neural networks (Forti & Tesi, 1995; Jin & Gupta, 1996), by regarding \(C\) as \(R^2\). Leaving minor details aside, we find that our class of activation functions is more general than theirs, but the conditions for the connection weight matrix are stricter.

For the condition of Theorem 1, it is necessary to choose suitable \(P\) and \(\gamma\). As to \(\gamma\), since Eq. (8) is the matrix inequality involving this single variable, it is rather easy to choose \(\gamma\) by the trial and error. Once \(\gamma\) is fixed, Eq. (8) is just the Lyapunov equation with respect to the positive definite diagonal matrix \(P\), and solutions of this equation can be easily obtained. We note that, for the condition (Eq. (14)), the above discussion similarly holds.

According to Theorem 1 and Theorem 2, it follows that there are classes of discrete-time and continuous-time complex-valued recurrent neural networks whose equilibrium point is identical and globally asymptotically stable. This fact is summarized in the following theorem.

**Theorem 4.** Suppose that \(f \in F^\infty\). If the connection weight matrix \(W\) satisfies the condition that \(L^{-1} - |W|\) is a nonsingular M-matrix, then there exist a discrete-time and a continuous-time complex-valued recurrent neural networks whose unique equilibrium points coincide with each other and are globally asymptotically stable.

**Proof:** It is easy to see that the networks (Eqs. (1) and (2)) can share the same equilibrium point. Suppose that the matrix \(X = L^{-1} - |W|\) is a nonsingular M-matrix. The matrix \(D X = D L^{-1} - |D W|\) is also a nonsingular M-matrix. Furthermore, there exists a positive definite diagonal matrix \(P\) such that \(L^{-1} P L^{-1} - |P| |W| > 0\) and \(P D X + (D X)^2 P > 0\) (Araki, 1974; 1977). Hence, from Corollary 1 and Theorem 2, the networks (Eqs. (1) and (2)) have a common equilibrium point and it is globally asymptotically stable. (Q. E. D.)
Under the condition of Theorem 4, the networks given by

\[ u[k + 1] = Wf(u[k]) + b, \]  \hfill (24)

\[
\frac{du(t)}{dt} = -Du(t) + DWf(u(t)) + Db
\]  \hfill (25)

where \( D \in \mathbb{R}^{n \times n} \) is an arbitrary positive definite diagonal matrix, are those mentioned in the theorem. The advantage of Theorem 4 lies in the point that both the discrete-time and continuous-time neural networks can be made simultaneously available depending on the applications of various optimization problems.

### Application to Convex Programming Problems

For the complex-valued recurrent neural networks (Eqs. (24), (25) and (2)), an application of the global asymptotical stability conditions to a convex programming problem is presented in this section. Consider the following convex programming problem whose variables are constrained by an arbitrary bounded, closed, and convex region in complex space.

\[
\min_{v} J(v) = \frac{1}{2} v^T M v + \text{Re}(q^T v)
\]

s. t. \( v_i \in K_i, \ (i = 1, 2, \ldots, n) \)  \hfill (26)

where \( M \in \mathbb{C}^{n \times n} \) is a positive definite Hermitian matrix, \( K_i \subset \mathbb{C} \) is a bounded, closed, and convex set including the origin of \( \mathbb{C} \), \( q = [q_1, q_2, \ldots, q_n]^T \in \mathbb{C}^n \), and \( v = [v_1, v_2, \ldots, v_n]^T \in \mathbb{C}^n \).

In general, for a point \( v^* \in \mathbb{C}^n \) to be an optimal solution of Eq. (26) it is necessary and sufficient that \( v^* \) is a fixed point of the map \( H(v) : \mathbb{C}^n \rightarrow \mathbb{C}^n \):

\[ H(v) = \Psi(v - \alpha M v - \alpha q) \ (\forall \ \alpha > 0) \]  \hfill (27)

where \( \Psi = [\psi_{K_i}(u_i), \psi_{K_i}(u_2), \ldots, \psi_{K_i}(u_n)]^T \) with \( \psi_{K_i}(u_i) \) being the Convex projection to \( K_i \) as given in Eq. (5). This can be understood in the following way. Because a complex plane can be regarded as two-dimensional real space, the problem (Eq. (26)) can be represented as the problem on \( \mathbb{R}^2 \). From the literature (Fukushima, 2001, pp. 203-241), this problem on \( \mathbb{R}^2 \) is further reduced to a fixed-point problem on \( \mathbb{R}^2 \) which is equivalent to the fixed-point problem for Eq. (27). Hence, the fixed point of Eq. (27) corresponds to the optimal solution of Eq. (26).

For the networks (Eqs. (24) and (25)), when specified as \( W = I_n - \alpha M, \ b = -\alpha q, \ f = \Psi, \) and \( D \) is an arbitrary positive definite diagonal matrix, their equilibrium points are the fixed points of Eq. (27). Therefore, if these networks are globally asymptotically stable, the trajectories of the networks converge to the global optimal solution. For that property, Theorem 4 gives the condition that there exists an \( \alpha > 0 \) such that \( I_n - \alpha M \) is a nonsingular M-matrix. One of the sufficient conditions for the above stability condition is provided as follows.

**Lemma 1.** If a Hermitian matrix \( M = \{m_{ij}\} \in \mathbb{C}^{n \times n} \) satisfies that

\[
|m_i| > \sum_{j \neq i} |m_{ij}| \quad (i = 1, 2, \ldots, n),
\]  \hfill (28)

and \( 0 < \alpha < 1/ \max |m_i| \), then the real matrix \( I_n - \alpha M \) is a nonsingular M-matrix.
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Proof: Suppose that \( M \) and \( \alpha \) satisfy the condition of Lemma 1. Since \( 1 - \alpha \mid m_{ij} \mid > 0 \), it holds that

\[
1 - \left| 1 - \alpha \mid m_{ij} \mid \right| = \alpha \mid m_{ij} \mid > \alpha \sum_{i \neq j} m_{ij} \mid.
\]

Hence, \( \text{tr} - \mid L_n - \alpha M \mid \) is a nonsingular M-matrix from the property of M-matrix (Araki, 1974; 1976).

\[ Q. \ E. \ D. \]

In a special case, where the constraint conditions of problem (Eq. (26)) are such that

\[
v_i \in K_i = \{ v_i \mid v_i \in C, \kappa_i > 0, \mid v_i \mid \leq \kappa_i \} \quad (i = 1, 2, \cdots, n), \tag{29}
\]

then relying on the following continuous-time complex-valued neural network, Theorem 3 gives the global optimal solution of the problem (Eqs. (26) and (29)),

\[
\begin{aligned}
\frac{du_i(t)}{dt} &= -Du_i(t) + (-M)f(u_i(t)) + (-q) \\
f_i(u_i) &= \varphi_i(|u_i|) \frac{u_i}{|u_i|} \in S, \quad \varphi_i(\cdot) \leq \kappa_i, \quad (i = 1, 2, \cdots, n). \tag{30}
\end{aligned}
\]

This can be shown in the following way similar to the case of real-valued neural networks. Consider the following energy function of the network (Eq. (30)),

\[
E(v) = \frac{1}{2} v^* M v + \text{Re}(q^* v) + \sum_{i=1}^{n} \int_{0}^{\rho} \varphi_i^{-1}(\rho) d\rho 
\]

where \( \varphi_i^{-1}(\cdot) \) is the inverse function of real-valued function \( \varphi_i(\cdot) \). Since \( \varphi_i(0) = 0 \) and \( \varphi_i(\cdot) \) is a monotone increasing function, the integral term of Eq. (31) is positive. Hence, the energy function (Eq. (31)) is lower bounded. With the polar-coordinate expression, define \( r_i = |u_i| \) and \( \theta_i = \arg(u_i) \), then the time derivative of Eq. (31) along the trajectory of Eq. (30) is calculated as follows:

\[
\frac{dE}{dt} \bigg|_{(30)} = -\sum_{i=1}^{n} d_i \left[ \frac{d\varphi_i}{dr_i} \left( \frac{dr_i}{dt} \right)^2 + r_i \varphi_i(r_i) \left( \frac{d\theta_i}{dt} \right)^2 \right] \leq 0.
\]

Furthermore, \( dE/dt \bigg|_{(30)} = 0 \) if and only if \( dr_i/dt = 0 \), \( d\theta_i/dt = 0 \). Hence the network (Eq. (30)) behaves so as to decrease the value of Eq. (31). Now because \( M \) is a positive definite Hermitian matrix, the connection weight matrix of Eq. (30) satisfies the condition (Eq. (19)) of Theorem 3. Therefore, the trajectories of Eq. (30) converge to the unique and globally asymptotically stable equilibrium point independently of initial conditions. Moreover, this equilibrium point is also the minimum point of the energy function (Eq. (31)). The energy function (Eq. (31)) is the objective function of the problem, which is translated from the convex programming problem (Eqs. (26) and (29)) to the unconstrained minimization problem with the nonlinear function term \( \int_{0}^{\rho} \varphi_i^{-1}(\rho) d\rho \). This third term of Eq. (31) and \( d_i > 0 \) \( (i = 1, 2, \cdots, n) \) are the penalty term and the penalty parameters, respectively. By suitably adjusting \( d_i > 0 \) \( (i = 1, 2, \cdots, n) \), the global optimal solution of problem (Eqs. (26) and (29)) can be approximated by the minimum of energy function (Eq. (31)). In this way, the complex-valued neural network (Eq. (30)) yields the approximate solution of the convex programming problem (Eqs. (26) and (29)).

Generally, real-valued recurrent neural networks can also solve this problem (Xia & Wang 2004). In this case, however, problem formulations may become much more complex depending on constraints. On the other hand,
with the complex-valued recurrent neural networks, it is sometimes possible to design and solve the problems much easier than the case of real-valued counterparts. Such a case can be shown in the next section.

**NUMERICAL EXAMPLE**

To demonstrate applications of the complex-valued neural networks (Eqs. (24), (25), and (30)) to solve convex programming problems, numerical examples are shown in this section.

**Example 1**
Consider the problem (Eq. (26)) whose parameters and constraints are set as follows:

\[
M = \begin{bmatrix}
6.0 & -4.0 + j2.0 \\
-4.0 - j2.0 & 7.0 \\
\end{bmatrix},
\]

\[
q = (6.0 + j9.0 \quad j10.0)^T,
\]

\[K_1 = \{ v_1 \in \mathbb{C} \mid |v_1| \leq 1, \arg v_1 \leq \pi / 6 \}, \]

\[K_2 = \{ v_2 \in \mathbb{C} \mid |v_2| \leq 1, \arg v_2 \leq \pi / 6 \}, \]

where \( \arg v_i \) denotes the argument of \( v_i \in \mathbb{C} \), and \( j \) is the imaginary unit. In this example, the convex set \( K \) is a bounded sector region on the complex plane, and the following projection function is available for the activation functions.

\[
\psi_{K_i}(u_i) = g_2(g_1(u_i))
\]

where \( \mu = \tan(\pi / 6) \), \( \sigma_1 = \mu^* \text{Re}(u_i) + |\text{Im}(u_i)| \), \( \sigma_2 = \mu^* \text{Re}(u_i) - |\text{Im}(u_i)| \), and

\[
g_1(u_i) = \frac{1}{1 + \mu^2} \left[ \mu \left( \max \{0, \sigma_1\} + \max \{0, \sigma_2\} \right) + j \text{sgn}(\text{Im}(u_i)) \left( \max \{0, \sigma_1\} - \mu^2 \max \{0, \sigma_2\} \right) \right],
\]

\[
g_2(u_i) = \min(1, |u_i|) \frac{u_i}{|u_i|}.
\]

It can be shown that the matrix \( M \) satisfies the condition of Lemma 1. Parameter values and the activation functions of the complex-valued recurrent neural networks (Eqs. (24) and (25)) can be set as \( \alpha = 0.125, D = \text{diag} \{2.0, 5.0\} \)

\[
W = (I - \alpha M), b = -\alpha q, \text{ and } f_1 \text{ and } f_2 \text{ are given by Eq. (32)}. \]

Note that the global optimal solution of this example is known as \((v^\text{opt}_1, v^\text{opt}_2) = (0.48 - j0.28, 0.87 - j0.50)\). As a result of simulations, both the networks (Eqs. (24) and (25)) converge towards the point \((v^\text{opt}_1, v^\text{opt}_2) = (0.477 - j0.275, 0.866 - j0.500)\), showing that the global optimal solution has been obtained. The output trajectories of neural networks (Eqs. (24) and (25)) converging to the global optimum point from an arbitrary taken initial conditions are shown in Figure 1.

**Example 2**
Next, we demonstrate an application of the network (Eq. (30)) to the problem (Eq. (26)) with constraint condition (Eq. (29)). Consider the problem (Eqs. (26) and (29)) whose parameters and constraints are set as follows:
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\[
M = \begin{pmatrix}
3.0 & 1.0 - j1.0 \\
1.0 + j1.0 & 4.0
\end{pmatrix}
\]

\[
q = (15 + j5.0, 20 + j20)
\]

\[\kappa_1 = 1.0, \kappa_2 = 1.0\]

In this example, the convex set \(K_i\) is a unit circle region on the complex plane. Parameter values and the activation functions of the complex-valued recurrent neural network (Eq. (30)) can be set as, \(D = \text{diag}(0.01, 0.01), W = -M, b = -q,\) and \(f_1\) and \(f_2\) are given by Eq. (7). Because \(W\) is a Hermitian matrix and its eigenvalues are -2.0 and -5.0, this network satisfies the condition of Theorem 3. Therefore, it has unique and globally asymptotically stable equilibrium point. Note that the global optimal solution of this example is known as \(\left( v_1^{opt}, v_2^{opt} \right) = (0.94 + j0.35, 0.72 + j0.69)\). As a result of simulations, the network (Eq. (30)) converges towards the point, \(\left( v_1^{opt}, v_2^{opt} \right) = (0.938 + j0.347, 0.720 + j0.694)\) showing that the global optimal solution has been obtained. The output trajectories of the variable \(v_1\) of neural network (Eq. (30)) converging to the global optimum point with some arbitrary initial conditions are shown in Figure 2.

CONCLUSION

In this chapter, the global asymptotic stability conditions for discrete-time and continuous-time complex-valued recurrent neural networks are presented. To derive the stability conditions, classes of complex-valued functions suitable for the activation functions are defined, and Lyapunov function method is used. According to the derived stability conditions, there are classes of discrete-time and continuous-time complex-valued networks whose equilibrium points coincide and are globally asymptotically stable. Furthermore, these networks are shown to be successfully applicable to solving convex programming problems with nonlinear, bounded, closed, and convex constraints.

**Figure 1.** Output trajectories of neural networks (Eqs. (24) and (25)) converging to the optimum point in complex plane. (a) Output trajectories of \(v_1\), (b) Output trajectories of \(v_2\).
Compared with real-valued recurrent neural networks, complex-valued neural networks possess deep potential for a wide variety of applications in mathematical optimization field. This is because of the added freedom in choosing activation functions other than the connection weight matrix.

**FUTURE RESEARCH DIRECTIONS**

In technological world, there are many occasions where one is often required to deal with complex-valued variables, especially in analysis and design of physical systems such as optical or electronic networks and appliances. Handling them optimally according to a given criterion or cost function so that it yields sole extreme value leads to an attempt explored in this chapter: solving global optimization problems with complex-valued recurrent neural networks. The attempt is still at a primary stage and much work is to be done. Here are some directions towards completion as a comprehensive optimization tool.

1. *Problem Formulations*
   It is first noted that in general what can be done by complex-valued recurrent neural networks can also be essentially done by real-valued recurrent neural networks or any other problem solvers in real number field. There comes the importance of problem formulations. Once a task of optimization with regard to some physical systems is given, it is recommended to examine first if one could take advantage of using complex-valued networks, compared to real-valued counterparts. Studying such problem formulations is an elaborate but challenging theme.

2. *Choice of Activation Functions*
   One of the strengths of complex-valued recurrent neural networks lies in the fact that their activation functions, complex-valued functions with complex-valued arguments, have much more freedom of choice than real-valued counterparts whose tuning ability hinges solely on the weight matrices. As demonstrated in the example of this chapter, a certain activation function may fit in a given optimization problem. This gives rise to matching between optimization problems and suitable activation functions with suitable parameters. The task appears to be little tricky,
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but accumulated case-studies would help to consolidate a comprehensive way towards finding the fittest.

3. Digital Implementation

Recurrent Neural networks are usually realized in computers, while given global optimization problem is used to be formulated in continuous-time framework, and the fit neural network as its solver is at first designed in the same framework. This entails digital implementation of designed neural network in such a way that the global equilibrium and its stability are preserved under discretizations. In the presented chapter, a possible approach to making this happen is suggested, but more works are anticipated about digital implementation.

REFERENCES


Global Stability Analysis for Complex-Valued Recurrent Neural Networks


ADDITIONAL READINGS


APPENDIX: PROOF OF PROPERTY 2

Because the relation is obvious in case of \( x_i = y_i \), consider the case of \( x_i \neq y_i \). With representing complex numbers as \( x_i = r_i e^{j \theta_i}, y_i = r_i e^{j \theta_i} \), Property 2 can be proved as follows.

(i) If \( r_1 \neq r_2 \), it is satisfied that

\[
0 < \frac{\varphi(r_1) - \varphi(r_2)}{r_1 - r_2} \leq l_1 < +\infty.
\]

By multiplying the inverse of this both side by \( \left[ \varphi(r_1) - \varphi(r_2) \right]^T \), we have

\[
l_1^{-1} \left[ \varphi(r_1) - \varphi(r_2) \right] \leq \left[ \varphi(r_1) - \varphi(r_2) \right] (r_1 - r_2).
\] (33)

Notice that \( \varphi(0) = 0 \) and hence \( l_1^{-1} \varphi(r_k) \leq r_k \), \( (k = 1, 2) \). The left hand side(LHS) of the inequality of Property 2 is calculated as follows.

\[
\text{Re} \left\{ \left[ x_i - y_i, f_i(x_i) - f_i(y_i) \right] \right\} = r_1 \varphi(r_1) + r_2 \varphi(r_2) - \{ r_1 \varphi(r_2) + r_2 \varphi(r_1) \} \cos(\theta_1 - \theta_2)
\]

\[
= \{ \varphi(r_1) - \varphi(r_2) \} (r_1 - r_2) + \{ r_1 \varphi(r_2) + r_2 \varphi(r_1) \} \{ 1 - \cos(\theta_1 - \theta_2) \}.
\]

From Eq. (33), and \( 1 - \cos(\theta_1 - \theta_2) > 0 \),

\[
\text{Re} \left\{ \left[ x_i - y_i, f_i(x_i) - f_i(y_i) \right] \right\} \geq l_1^{-1} \left[ \varphi(r_1) - \varphi(r_2) \right]^T + \{ r_1 \varphi(r_2) + r_2 \varphi(r_1) \} \{ 1 - \cos(\theta_1 - \theta_2) \}
\]

\[
= l_1^{-1} \left[ l_1^{-1} \varphi(r_1) e^{j \theta_1} - \varphi(r_2) e^{j \theta_2} \right]^T + \left[ \{ r_1 - l_1^{-1} \varphi(r_1) \} \varphi(r_2) + \{ r_2 - l_1^{-1} \varphi(r_2) \} \varphi(r_1) \right] \{ 1 - \cos(\theta_1 - \theta_2) \}
\]

\[
\geq l_1^{-1} \left| f_i(x_i) - f_i(y_i) \right|^2.
\]

(ii) If \( r_1 = r_2 \) and \( \theta_1 \neq \theta_2 \), from \( l_1^{-1} \varphi(r_k) \leq r_k \), \( (k = 1, 2) \), the LHS of the inequality is evaluated as follows.

\[
\text{Re} \left\{ \left[ x_i - y_i, f_i(x_i) - f_i(y_i) \right] \right\} = r_1 \varphi(r_1) e^{j \theta_1} - e^{j \theta_2} \left| e^{j \theta_1} - e^{j \theta_2} \right|^2
\]

\[
\geq l_1^{-1} \varphi(r_1) \left| e^{j \theta_1} - \varphi(r_1) e^{j \theta_2} \right|^2
\]

\[
= l_1^{-1} \left| \varphi(r_1) e^{j \theta_1} - \varphi(r_1) e^{j \theta_2} \right|^2.
\]

Since, \( r_1 = r_2 \), we have

\[
\text{Re} \left\{ \left[ x_i - y_i, f_i(x_i) - f_i(y_i) \right] \right\} \geq l_1^{-1} \left| \varphi(r_1) e^{j \theta_1} - \varphi(r_2) e^{j \theta_2} \right|^2
\]

\[
= l_1^{-1} \left| f_i(x_i) - f_i(y_i) \right|^2.
\]

Hence, Property 2 is proved. (Q. E. D.)